

An introduction to the volume conjecture

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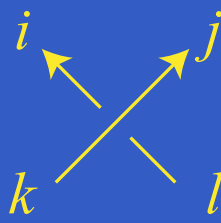
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$$\blacksquare J_N(K; q) \approx \sum_{\substack{0 \leq i, j, k, l \leq N-1 \\ i, j, k, l: \text{labelings}}} \left(\prod_{\text{crossings}} R_{kl}^{ij} \right)$$


The diagram shows a crossing of two strands. The top-left strand is labeled i , the top-right strand is labeled j , the bottom-left strand is labeled k , and the bottom-right strand is labeled l . The strands i and j cross over the strands k and l .

R-matrix



$$R_{kl}^{ij} := \sum_{m=0}^{\min(N-1-i, j)} \delta_{l, i+m} \delta_{k, j-m} \\ \times (q^{1/2} - q^{-1/2})^m \frac{[i+m]! [N-1+m-j]!}{[i]! [m]! [N-1-j]!} \\ \times q^{(i-(N-1)/2)(j-(N-1)/2) - m(i-j)/2 - m(m+1)/4},$$

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$$[k] := \frac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}}.$$

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Volume Conjecture (R. Kashaev, J. Murakami+HM)

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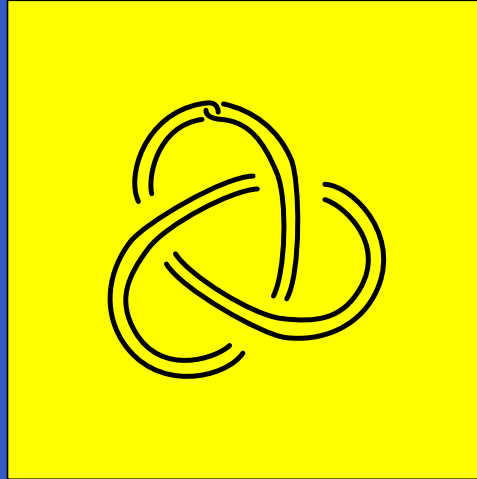
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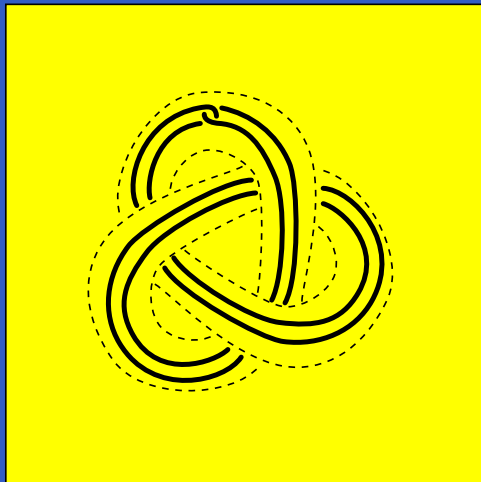
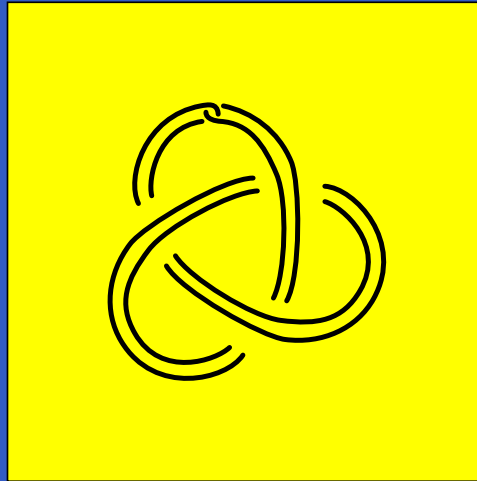
- H_i : hyperbolic piece,
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$$\text{Vol}(S^3 \setminus K) := \sum_{H_i: \text{hyperbolic pieces}} \text{Vol}(H_i)$$

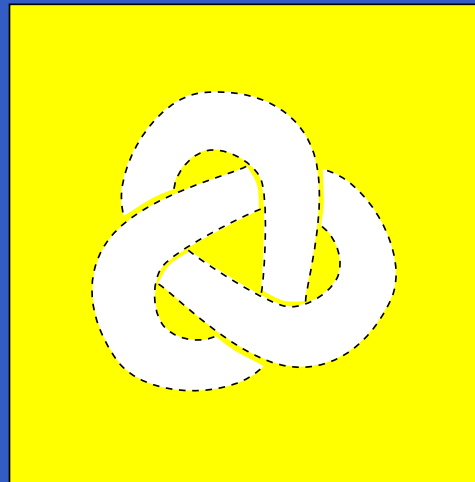
Jaco–Shalen–Johanson decomposition



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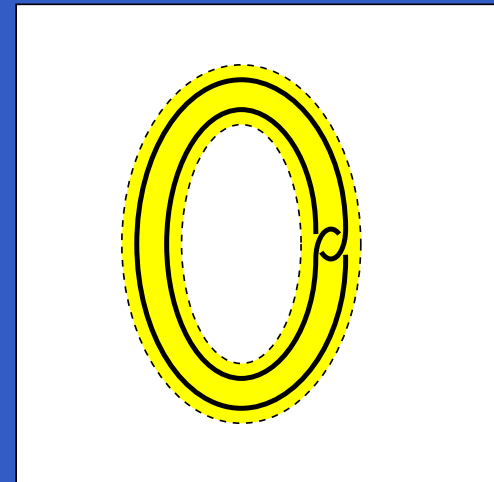


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Seifert fibered

∪



hyperbolic

Figure-eight knot

- $E :=$ figure-eight knot

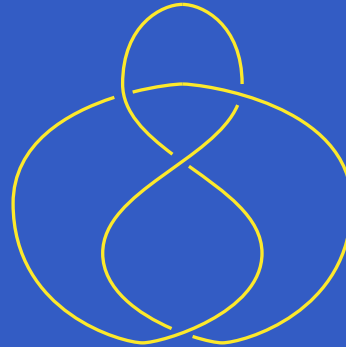
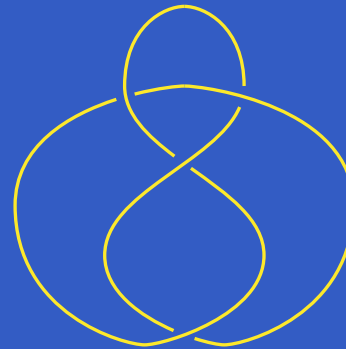


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(K. Habiro and T. Le)

$$J_N(E; q) = \sum_{k=0}^{N-1} \prod_{j=1}^k \left(q^{(N+j)/2} - q^{-(N+j)/2} \right) \left(q^{(N-j)/2} - q^{-(N-j)/2} \right).$$

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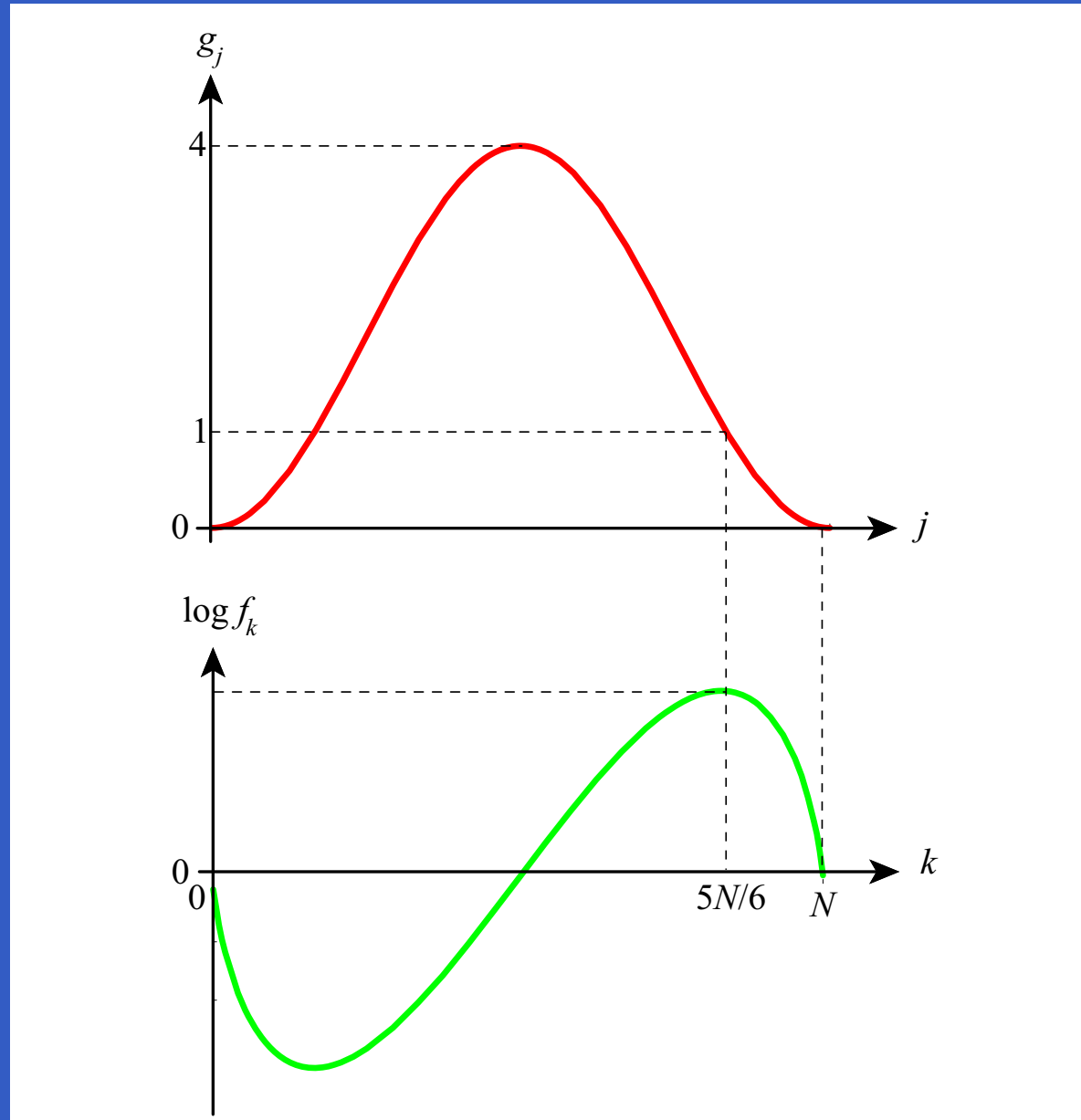
We also put

$$f_k := \prod_{j=1}^k g_j$$

so that

$$J_N(E; \exp(2\pi\sqrt{-1}/N)) = \sum_{k=1}^{N-1} f_k$$

Graphs of g_j and $\log f_k$



Proof of the Volume Conjecture for by T. Ekhholm (1)

$$f_{5N/6} < \sum_{k=0}^{N-1} f_k < N f_{5N/6} .$$
$$\parallel$$
$$J_N(E; \exp(2\pi\sqrt{-1}/N))$$

Proof of the Volume Conjecture for \mathfrak{S} by T. Ekhholm (1)

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We use

$$\lim_{N \rightarrow \infty} \frac{\log N}{N} = 0 .$$

Proof of the Volume Conjecture for \mathfrak{S} by T. Ekhholm (2)

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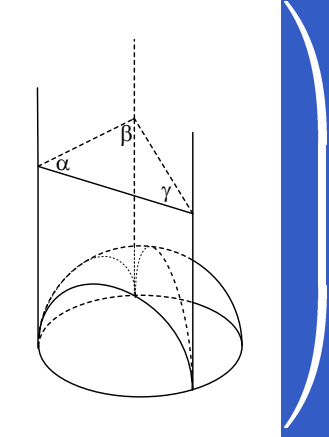
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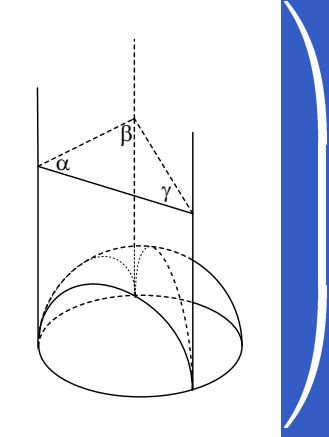
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$$= -\frac{2}{\pi} L(5\pi/6) = \frac{6L(\pi/3)}{2\pi}.$$

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A diagram showing a spherical cap on a sphere. A vertical dashed line represents the axis of symmetry. A horizontal dashed line represents the base of the cap. A solid line from the center of the sphere to the edge of the cap is labeled alpha. A solid line from the center of the cap to the edge is labeled beta. A solid line from the center of the sphere to the center of the cap is labeled gamma.

$$\Rightarrow \lim_{N \rightarrow \infty} \frac{\log J_N(E; \exp(2\pi\sqrt{-1}/N))}{N} = \frac{6L(\pi/3)}{2\pi}$$

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- The complement of \mathfrak{S} can be decomposed into two **regular** ideal hyperbolic tetrahedra. So the Volume Conjecture is true for \mathfrak{S} .

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- Borromean rings (S. Garoufalidis and T. Lê).

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We could expect

$$\frac{\log |J(K; \exp(2\pi\sqrt{-1}/N))|}{N} \underset{N \rightarrow \infty}{\sim} \frac{\log F_\lambda}{N}.$$

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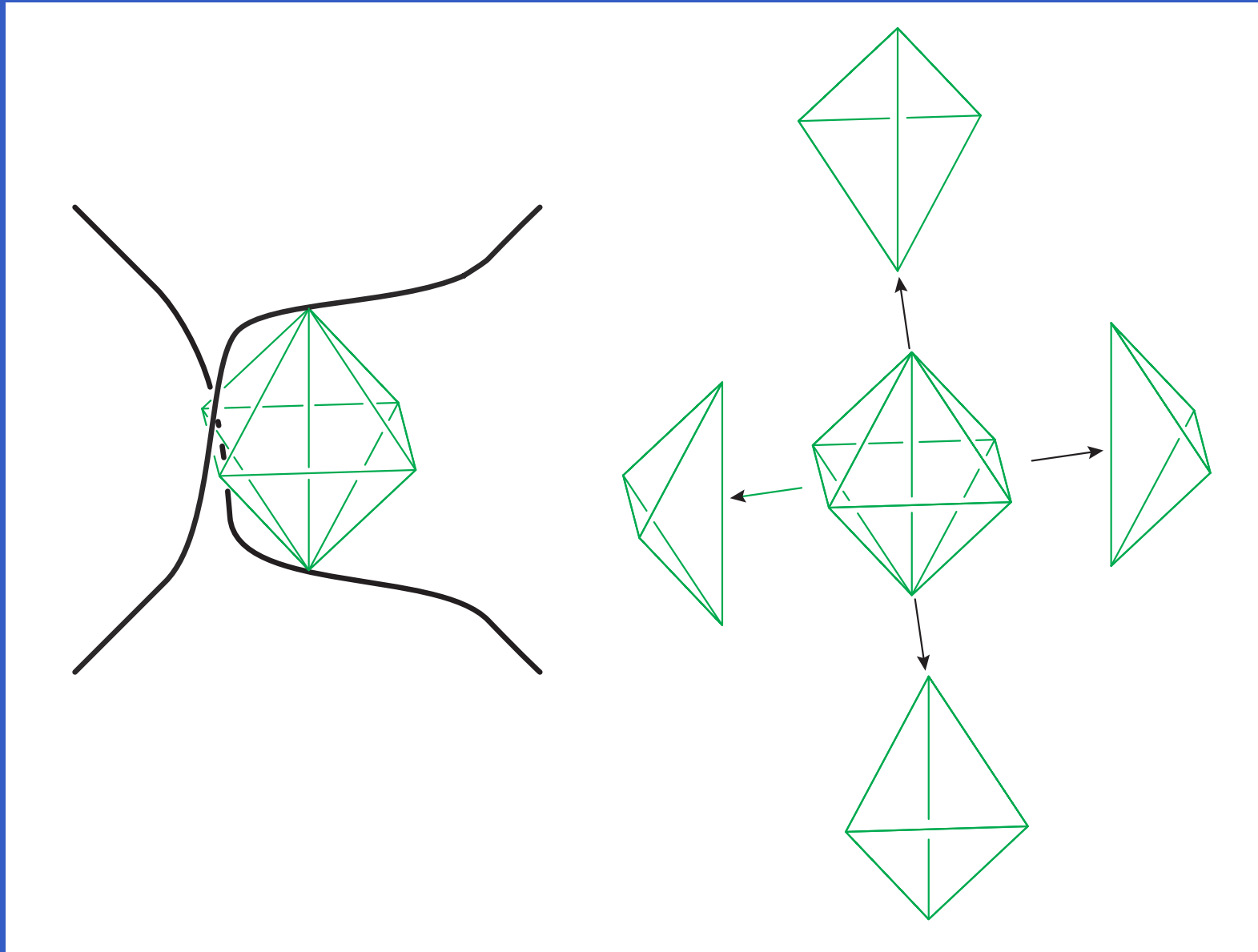
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- The labels would correspond to the complete hyperbolic structure of the knot complement (if the knot has a complete hyperbolic structure).

Four tetrahedra around a vertex



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- ⇒ We have several solutions corresponding to the stationary points.

The problem is which solution gives the 'biggest one'.

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$$2\pi \lim_{N \rightarrow \infty} \frac{\log J_N \left(K; \exp(2\pi\sqrt{-1}/N) \right)}{N} \\ = \text{Vol}(S^3 \setminus K) + \sqrt{-1} \text{CS}(S^3 \setminus K).$$

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- One may regard this as a definition of the Chern–Simons invariant for a (non-hyperbolic) knot.

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to be continued...