A generalization of the volume conjecture

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Generalization to three-manifolds

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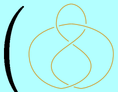
$$2\pi \lim_{N\to\infty} \frac{\log J_N\left(K; \exp(2\pi\sqrt{-1}/N)\right)}{N}$$
$$= \mathrm{Vol}(S^3 \setminus K) + \sqrt{-1}\,\mathrm{CS}(S^3 \setminus K).$$

$$2\pi \lim_{N\to\infty} \frac{\log J_N\left(K; \exp(\ a\ /N)\right)}{N} = ?$$

to be continued...

Result on 

Theorem. (Y. Yokota, HM)

The limit $\displaystyle\lim_{N\to\infty} \frac{1}{N}\log J_N\big($  $;\exp(a/N)\big)$

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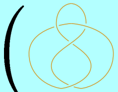
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- $$H(u) := (u + 2\pi\sqrt{-1}) \times \lim_{N\to\infty} \frac{\log J_N\left(K ; \exp((u+2\pi\sqrt{-1})/N)\right)}{N}$$

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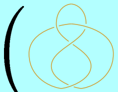

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  for small u.
- $a = u + 2\pi\sqrt{-1}$.

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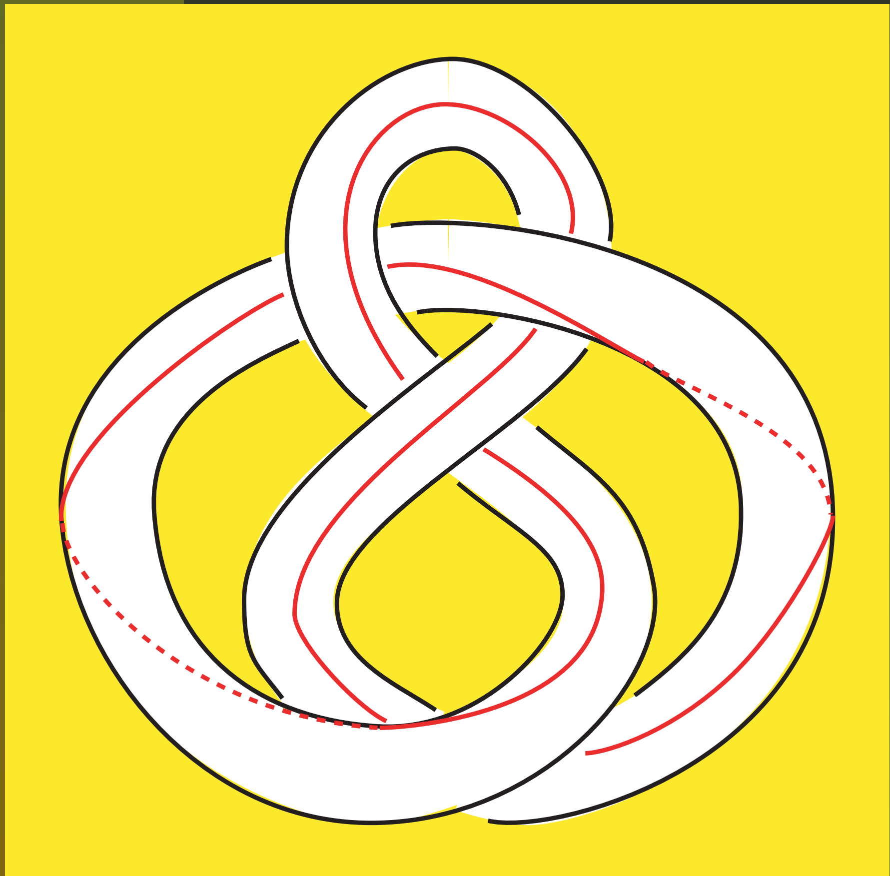
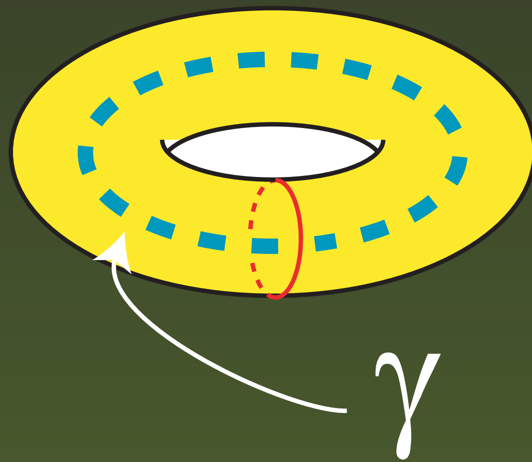
- Chern–Simons invariant can also be obtained (later).

Dehn surgery

 $\cup$ 

Identifying two red circles, we get the $2/1$-surgery.

Analytic version of MMR conjecture

Theorem. **(analytic version of the Melvin–Morton–Rozansky conjecture)** (S. Garoufalidis and T. Le)

For any knot K,

$$\lim_{N\to\infty} J_N(K;\exp(a/N)) = \frac{1}{\Delta(K;\exp a)}$$

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- We do *not* divide by N in the limit.

Result on torus knots

Theorem. (HM)

$$\lim_{N\to\infty} \frac{\log J_N\left(T(\alpha,\beta); \exp(a/N)\right)}{N} = \pi\sqrt{-1} + \frac{\pi^2}{\alpha\beta a} - \frac{\alpha\beta a}{4}.$$

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- $|a|>\dfrac{2\pi}{\alpha\beta}$, $\mathrm{Re}(a)<0$, and $\mathrm{Im}(a)>0$.

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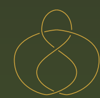
$$\begin{cases} \text{quantity that gives Vol} & \text{for } \text{(knot)}, \\ 0 & \text{if } |a| \text{ is small}, \\ \text{strange quantity} & \text{for torus knots.} \end{cases}$$



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$\Rightarrow$ Can we relate $\heartsuit$ to a kind of volume?

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$$\rho : \begin{cases} \text{meridian} & \mapsto \begin{pmatrix} \exp(u/2) & * \\ 0 & \exp(-u/2) \end{pmatrix}, \\ \text{longitude} & \mapsto \begin{pmatrix} \exp(v/2) & * \\ 0 & \exp(-v/2) \end{pmatrix}. \end{cases}$$

Volume function

- $V(K;u)$ satisfies Schläfli's formula:

$$dV(K;u)$$
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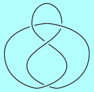
$\Rightarrow$ The conjecture is true for

- ,
- any knot with small $|a|$,
- torus knots.

Chern–Simons invariant?

Theorem. (Y. Yokota, HM)

Vol( $_u$)

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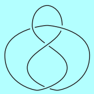
CS( $_u$)

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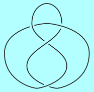
$\mathrm{Vol}($  $_u)$

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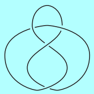
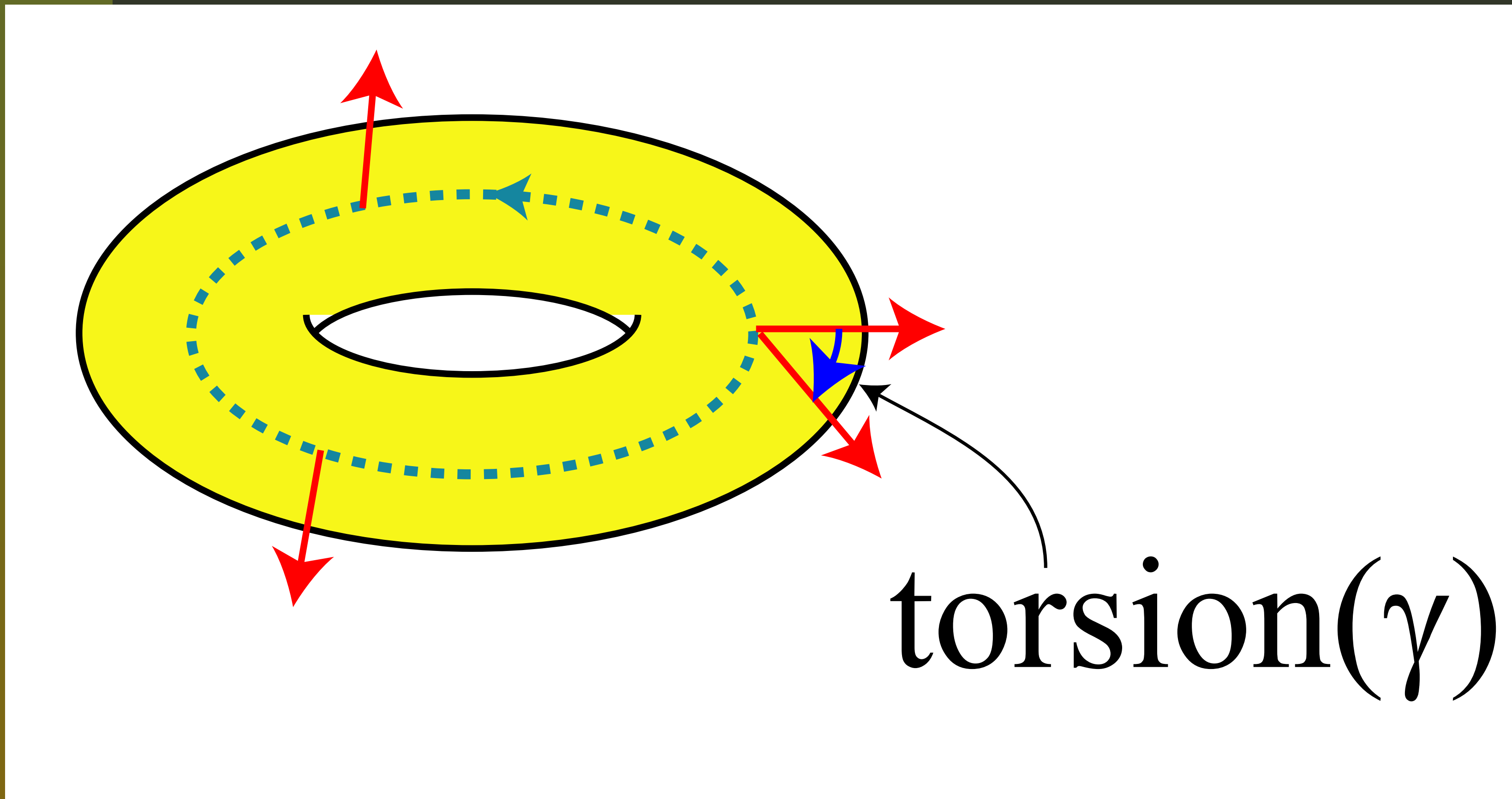
$\mathrm{CS}($  $_u)$

$$= \operatorname{Re} H(u) - \pi \operatorname{Im}(u) - \frac{1}{4}\operatorname{Re}(uv) - \frac{\pi}{2}\,\mathrm{torsion}(\gamma)$$

-  $_u$ is the 3-manifold obtained by Dehn surgery along  corresponding to u.

Torsion



$\mathrm{torsion}(\gamma)$ measures how much the normal vector is twisted when it travels along γ in  $_u$.

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